

Limiti-Taylor-Serie1

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Gennaio 2009- Studocu

Determinate per quali valori di $\alpha > 0$ risulta finito il limite

$$\lim_{x \rightarrow 0} \frac{(1+2x)^x - 1 - \alpha x^\alpha}{x^{1+\alpha}}$$

e calcolatelo.

$$\lim_{x \rightarrow 0} \frac{(1+2x)^x - 1 - \alpha x^\alpha}{x^{1+\alpha}}$$

$$(1+2x)^x = e^{x \cdot \ln(1+2x)}$$

Arriviamo con lo sviluppo di Taylor all'ordine 3

$$\ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3$$

$$\begin{aligned} \ln(1+2x) &= 2x - \frac{1}{2}(2x)^2 + \frac{1}{3}(2x)^3 = \\ &= 2x - 2x^2 + \frac{8}{3}x^3 + o(x^3) \end{aligned}$$

$$x \cdot \ln(1+2x) = 2x^2 - 2x^3 + o(x^3)$$

$$e^t - 1 = t + \frac{1}{2}t^2 + \frac{1}{6}t^3$$

$$e^{x \ln(1+2x)} - 1 = (1+2x)^x$$

$$= 2x^2 - 2x^3 + o(x^3)$$

$$\begin{aligned}
&= e^{2x^2 - 2x^3 + o(x^3)} - 1 = \\
&= (2x^2 - 2x^3) + \frac{1}{2} (2x^2 - 2x^3)^2 + \frac{1}{6} (2x^2 - 2x^3)^3 + o(x^3) \\
&= 2x^2 - 2x^3 + \frac{1}{2} (2x^4 - \dots) + o(x^3)
\end{aligned}$$

Riscriviamo il limite:

$$\begin{aligned}
&\lim_{x \rightarrow 0} \frac{(1+2x)^x - 1 - dx^\alpha}{x^{1+d}} \\
&= \lim_{x \rightarrow 0} \frac{2x^2 - 2x^3 - dx^\alpha + o(x^3)}{x^{1+d}}
\end{aligned}$$

Il termine dominante al numeratore è x^2

Notiamo che se $\alpha = 2$ al numeratore tale termine scompare

$$\begin{aligned}
&\lim_{x \rightarrow 0} \frac{2x^2 - 2x^3 - 2x^2 + o(x^3)}{x^3} \\
&= \lim_{x \rightarrow 0} \frac{\cancel{2x^2} - 2x^3 - \cancel{2x^2} + o(x^3)}{x^3} = 2
\end{aligned}$$

se $\alpha > 2 \rightarrow$

$$\lim_{x \rightarrow 0^+} \frac{2x^2 + o(x^2)}{x^{\alpha+1}} = +\infty$$

se $\alpha < 2$:

$$\lim_{x \rightarrow 0^+} \frac{-dx^\alpha + o(x^\alpha)}{x^{\alpha+1}} = -\infty$$

PROBLEMA 3

Determinate per quali valori di $\alpha \in \mathbb{R}$ converge la serie

$$\sum_{n=1}^{\infty} \left| \operatorname{sen}\left(\frac{2}{n}\right) + e^{-1/n} \log\left(\frac{n-2}{n}\right) \right|^{\alpha}.$$

posto $x = \frac{1}{n}$

$$f(x) = \operatorname{sen} 2x + e^{-x} \cdot \log(1-2x)$$

Arriviamo con Taylor fino all'ordine 3

$$\operatorname{sen} t = t - \frac{1}{6} t^3$$

$$\begin{aligned} \operatorname{sen}(2x) &= 2x - \frac{1}{6} (2x)^3 + o(x^3) \\ &= 2x - \frac{4}{3} x^3 + o(x^3) \end{aligned}$$

$$e^t = 1 + t + \frac{1}{2} t^2 + \frac{1}{6} t^3$$

$$\begin{aligned} e^{-x} &= 1 + (-x) + \frac{1}{2} (-x)^2 + \frac{1}{6} (-x)^3 + o(x^3) \\ &= 1 - x + \frac{1}{2} x^2 - \frac{1}{6} x^3 + o(x^3) \end{aligned}$$

$$\log(1+t) = t - \frac{1}{2} t^2 + \frac{1}{3} t^3 + o(t^3)$$

$$\begin{aligned} \log(1-2x) &= -2x - \frac{1}{2} (-2x)^2 + \frac{1}{3} (-2x)^3 + o(x^3) \\ &= -2x - 2x^2 - \frac{8}{3} x^3 + o(x^3) \end{aligned}$$

$$e^{-x} \cdot \log(1-2x)$$

$$= \left(1 - x + \frac{1}{2} x^2 - \frac{1}{6} x^3\right) \cdot \left(-2x - 2x^2 - \frac{8}{3} x^3\right) + o(x^3)$$

$$\begin{aligned}
&= -2x - 2x^2 - \frac{8}{3}x^3 + 2x^2 + 2x^3 - x^3 + o(x^3) \\
&= -2x - \cancel{2x^2} - \frac{8}{3}x^3 + \cancel{2x^2} + 2x^3 - x^3 + o(x^3) \\
&= -2x - \frac{5}{3}x^3 + o(x^3)
\end{aligned}$$

$$f(x) = \boxed{\sin 2x} + \boxed{e^{-x}} \cdot \boxed{\log(1-2x)}$$

$$\begin{aligned}
f(x) &= 2x - \frac{4}{3}x^3 - 2x - \frac{5}{3}x^3 + o(x^3) \\
&= -3x^3 + o(x^3)
\end{aligned}$$

$$\begin{aligned}
&\lim_{x \rightarrow 0} |f(x)|^\alpha = \\
&= \lim_{x \rightarrow 0} \left| \sin 2x + e^{-x} \cdot \log(1-2x) \right|^\alpha \\
&\lim_{x \rightarrow 0} \left| -3x^3 + o(x^3) \right|^\alpha
\end{aligned}$$

Allora per quanto riguarda la nostra serie

$$\begin{aligned}
&\leq \left| \sin \frac{2}{m} + e^{-1/m} \cdot \log\left(1 - \frac{2}{m}\right) \right|^\alpha \\
&\sim \leq \left| -\frac{3}{m^3} \right|^\alpha = 3 \leq \left(\frac{1}{m^3}\right)^\alpha = 3 \leq \frac{1}{m^{3\alpha}}
\end{aligned}$$

$$3\alpha > 1 \rightarrow \alpha > \frac{1}{3}$$

Per il criterio del confronto con le serie armonica generalizzata